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TECHNICAL NOTE

A parametric representation of diffusive enhancement in a confined oscillatory flow

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INTRODUCTION

It has been known for some time that axial dispersion in an oscillatory flow in a confined passageway is much higher than that attainable by pure molecular diffusion. Variations of velocities in the cross-sectional area, coupled to the radially-distributed concentration field, leads to enhancement of axial dispersion. As a measure of the enhancement, the concept of effective diffusivity has been proposed, which represents the combined contributions of convective activities and of molecular diffusion. Taylor [1] pioneered on the subject of the diffusive spreading of a substance, and a considerable amount of work [2–5] has been compiled to address this issue of augmentation of axial dispersion in oscillating laminar flows.

For flows in a circular pipe or in a two-dimensional channel, Watson [2] and Kurzweg [6] derived the dimensionless effective diffusivity R that was governed by three non-dimensional quantities. Those quantities were the Schmidt number, the Womersley number α_h corresponding to a dimensionless frequency, and a dimensionless amplitude ε_1 formed by a representative length scale and a tidal displacement. Watson [2] also showed that $R_1(\equiv R/\varepsilon_1)$, the ratio of dimensionless effective diffusivity to dimensionless amplitude, was a monotonically increasing function of α_h. On the contrary, this continuous increase of R_1, within the range of large α_h, has conflicted with such a physical understanding that the enhancement diminishes to zero for both very high and very low values of an appropriately defined dimensionless frequency with other conditions being fixed [3]. Thus, to overcome this point, Joshi [3] demonstrated that R/Pe^2, the ratio of dimensionless effective diffusivity to another dimensionless amplitude, approached zero for the large α_h, where the Peclet number Pe was consisted of a velocity amplitude, the molecular diffusivity and a radius. However, this mapping, R/Pe^2 vs α_h, also could not meet the physical understanding for the very small α_h, quasi-steady situation. These discrepancies of previous works are due to the improper parameters and the physical analysis using only two kinds of time scales, a radial molecular diffusion time and a cycle period by Joshi [3] and Kurzweg [7, 8].

This study aims to find proper parameters in both low and high frequencies, based on a characteristic time scale analysis. Furthermore, using these new parameters, it will be shown that not only the radial diffusion time but also the axial diffusion time should be introduced for the analysis and physical understanding of the phenomena. We shall demonstrate that the case of a uniform rectangular duct shares the same physics as that of a circular tube.

GOVERNING EQUATIONS AND THE DEFINITION OF THE EFFECTIVE DIFFUSIVITY

It will be assumed that a diffusing substance is a passive contaminant, of which the concentration is so small that the physical properties of the fluid and the diffusivity of the contaminant may be taken as constant. The flow is assumed to be entirely in the axial direction with a uniform rectangular duct of width $2a$ and height $2b$. The axial momentum equation for a fully developed laminar flow bounded by a pipe wall B_1 and an axis of symmetry B_2, can be written as:

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \quad \text{in } \Omega \tag{1}$$

with the imposed pressure gradient $\partial p/\partial z = -P\cos(\omega t)$. The boundary conditions are

$$w = 0 \quad \text{on} \quad B_1 \quad \text{and} \quad \partial w/\partial n = 0 \quad \text{on} \quad B_2$$

where $\partial/\partial n$ denotes the rate of change in the outward normal direction. The diffusion equation governing the concentration is

$$\frac{\partial \theta}{\partial t} + w\frac{\partial \theta}{\partial z} = \kappa\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) \quad \text{in } \Omega \tag{2}$$

with

$$\partial\theta/\partial n = 0 \quad \text{on} \quad B_1 \quad \text{and} \quad B_2$$

since the pipe is assumed to be impermeable. Here, the axial gradient of concentration is assumed as constant.

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NOMENCLATURE

A	cross-sectional area of the tube
a	half width of the tube
a_h	hydraulic radius
b	half height of the tube
B_1	tube wall
B_2	axis of symmetry
i	imaginary unit
K	effective diffusivity
n	outward normal direction
P	amplitude of pressure gradient
p	pressure
q	axial volume flux
R	dimensionless effective diffusivity
s	aspect ratio
t	time
V	tidal volume
w	axial velocity
x	cross-stream coordinate
y	cross-stream coordinate
z	axial coordinate.

Greek symbols

α_h	Womersley number
γ	axial concentration gradient
η	non-dimensional cross-stream coordinate
θ	concentration
κ	molecular diffusivity
ν	kinematic viscosity
ξ	non-dimensional cross-stream coordinate
ρ	density
σ	Schmidt number
χ	tidal displacement
Ω	domain
ω	angular frequency of oscillation.

Superscript

$-$	time average
$\sim$	complex conjugate.

Solutions of equations (1) and (2) appear as below:

$$w = Re[f(x,y)e^{i\omega t}] \tag{3}$$

$$\theta = -\gamma z + Re[\gamma g(x,y)e^{i\omega t}]. \tag{4}$$

The flux q of the contaminant gas at z = constant is

$$q = w\theta - \kappa\frac{\partial\theta}{\partial z}. \tag{5}$$

If the time averaged flux consisted of the convection and diffusion in the equation (5) can be expressed by the concept of effective diffusion, then the effective diffusivity K is defined as:

$$\overline{w\theta} - \kappa\frac{\partial\bar{\theta}}{\partial z} = -K\frac{\partial\bar{\theta}}{\partial z} \quad \text{in} \quad \Omega. \tag{6}$$

Integrating the equation (6) over Ω using equations (3) and (4) gives the dimensionless effective diffusivity R,

$$R \equiv \frac{K}{\kappa} - 1 = \frac{1}{2\kappa A}Re\left[\iint_\Omega (f\tilde{g})\,dx\,dy\right]. \tag{7}$$

Dimensionless variables are introduced as:

$$\xi = x/a \quad \eta = y/b \quad \sigma = \nu/\kappa \quad s = a/b.$$

$$\alpha_h = a_h\sqrt{(\omega/\nu)} \quad a_h = 2ab/(a+b)$$

$$F(x,y) = 1 + \frac{\rho\omega}{iP}f(x,y)$$

$$G(x,y) = 1 + \frac{\rho\omega^2}{P}g(x,y).$$

From Watson [2] and Kurzweg [6] the tidal displacement χ becomes

$$\chi = \frac{2P}{\rho\omega^2}\left|\iint_\Omega (F-1)\,d\xi\,d\eta\right|. \tag{8}$$

Non-dimensionalized governing equations are

$$\frac{\partial^2 F}{\partial\xi^2} + c_1\frac{\partial^2 F}{\partial\eta^2} = ic_2\alpha_h^2 F \quad \text{in} \quad \Omega \tag{9}$$

$$\frac{\partial^2 G}{\partial\xi^2} + c_1\frac{\partial^2 G}{\partial\eta^2} = ic_2\sigma\alpha_h^2(G-F) \quad \text{in} \quad \Omega \tag{10}$$

with the boundary conditions

$$F = 1 \quad \text{on} \quad B_1 \quad \text{and} \quad \partial F/\partial n = 0 \quad \text{on} \quad B_2$$

$$\partial G/\partial n = 0 \quad \text{on} \quad B_1 \quad \text{and} \quad B_2$$

where

$$c_1 = s^2 \quad c_2 = (1+s)^2/4.$$

RESULTS AND DISCUSSION

Solution

After equations (9) and (10) are split into the real and imaginary parts, the Galerkin method is applied to obtain solutions as follows:

$$F - 1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\cos(\pi_m\xi)\cos(\pi_n\eta) + i\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}\cos(\pi_m\xi)\cos(\pi_n\eta) \tag{11}$$

$$G = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\cos(\Pi_m\xi)\cos(\Pi_n\eta) + i\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}\cos(\Pi_m\xi)\cos(\Pi_n\eta). \tag{12}$$

The coefficients a_{mn}, b_{mn}, c_{mn}, d_{mn} and arguments in cosine functions are in the Appendix. From the equation (7) the dimensionless effective diffusivity R_1 is derived,

$$R_1 \equiv \frac{R}{V^2/a_h^6} = \frac{1}{8}\frac{s^2}{(1+s)^4}\sigma\alpha_h^2\iint_\Omega Re[H]\,d\xi\,d\eta \Bigg/ \left|\iint_\Omega (F-1)\,d\xi\,d\eta\right|^2 \tag{13}$$

where $H = i(F-1)(\tilde{G}-1)$ and V is tidal volume whose definition is given in the Appendix.

Comparison of the dimensionless effective diffusivity with that in a circular pipe

Figure 1 shows that trends of the dimensionless effective diffusivity R_1 are in good agreement with Watson's solution [2] for a circular pipe. As pointed out by Watson [2], R_1 steeply increases in slow oscillation (low α_h) more than in fast oscillation. It is necessary to describe the dimensionless effective diffusivity R in view of new parameters for explaining physically the behaviour of R at the high values of α_h.

Analysis using new parameters—new dimensionless frequency and amplitude

In order to introduce new parameters, several characteristic time scales are defined as:

$\tau_v = a_h^2/\nu$, $\tau_t = a_h^2/\kappa$:	transverse diffusion time scales,
$\tau_c = \omega^{-1}$:	convective time scale,
$\tau_l = \chi^2/\kappa$:	longitudinal diffusion time scale.

Dimensionless variables which have been defined hitherto can be expressed by the time scale ratios as follows:

$$\alpha_h = \sqrt{(\tau_v/\tau_c)} \quad \sigma = \tau_t/\tau_v \quad (\chi/a_h)^2 = \tau_l/\tau_t \equiv \varepsilon_2$$

$$\sigma\alpha_h^2 = \tau_t/\tau_c \equiv \varepsilon_3 \quad \omega\chi^2/\kappa = \tau_l/\tau_c = \varepsilon_2\varepsilon_3 \equiv \varepsilon_4.$$

Using the above parameters, a new expression for the dimensionless effective diffusivity is,

$$R_2 \equiv \frac{R}{\omega\chi^2/\kappa} = \frac{1}{8}\iint_\Omega Re[H]\,d\xi\,d\eta \Bigg/ \left|\iint_\Omega (F-1)\,d\xi\,d\eta\right|^2. \tag{14}$$

From equation (14),

$$R = \varepsilon_2\varepsilon_3 R_2 = \varepsilon_4 R_2. \tag{15}$$

Here the new parameters ε_3 and ε_4 can be thought as a dimensionless frequency and a dimensionless amplitude.

As clearly seen from the equation (15), R is just proportional to ε_4. The effect of ε_3 on R is not so simple that a slow oscillation should be distinguished from a fast oscillation. From Fig. 2, an increase of ε_3 acts to enhance the axial dispersion in a slow oscillation, but a further increase of ε_3 tends to reduce that in a fast oscillation when ε_4 and σ are kept constant. These results in Fig. 2, clearly demonstrate that enhancement of axial dispersion vanishes for both a quasi-steady situation and very high values of dimensionless frequency ε_3 if ε_4 is kept constant. Another effectiveness of new parameters ε_3 and ε_4 is also shown in Fig. 2, where two parameters α_h and σ are unified into one parameter ε_3 in the range of slow oscillation. Consequently, it is important to note that the key parameter ε_4 can help to bridge over discrepancies between mathematical descriptions and physical understanding.

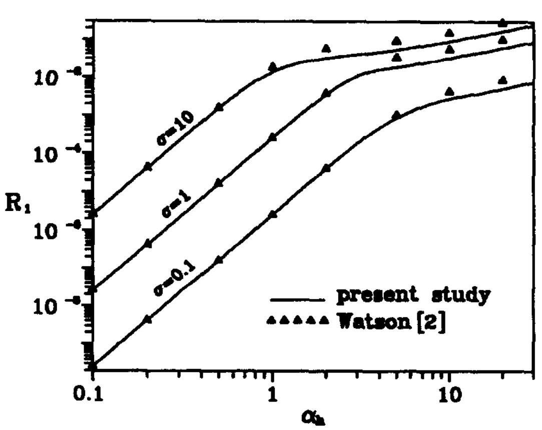

Fig. 1. The behavior of dimensionless effective diffusivity $R_1 (s = 1.0)$.

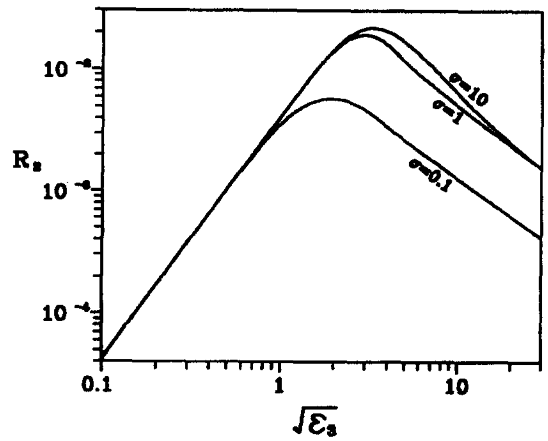

Fig. 2. The effect of ε_3 on the dimensionless effective diffusivity ($s = 1.0$).

In view of the characteristic time scale, the parameter ε_3 as the dimensionless frequency includes the concentration related scale rather than the momentum related one from the fact, $\alpha_h = a_h\sqrt{(\omega/\nu)}$ and $\sqrt{\varepsilon_3} = a_h\sqrt{(\omega/\kappa)}$. Therefore, ε_3 may be interpreted as whether or not the radial mixing, due to molecular diffusion occurs sufficiently during one half period of the oscillation, even though π is omitted in the definition of the convective time scale τ_c. For a further understanding of the physical meaning of the new dimensionless amplitude ε_4, the longitudinal length scales are introduced as:

$\delta_c = \chi$	convective length scale,
$\delta_\kappa = \sqrt{(\kappa/\omega)}$:	longitudinal diffusion length scale.

Here, the length scale δ_κ has a similar nature with the penetration depth of viscous wave in Stokes's 2nd problem (Schlitching [9]). Accordingly, δ_κ can be interpreted as the axial penetration depth of concentration, which signifies the axial distance of diffusion during a half period. With these two longitudinal length scales, ε_4 has a new expression,

$$\varepsilon_4 = (\delta_c/\delta_\kappa)^2 = \tau_l/\tau_c.$$

Therefore, the dimensionless parameter ε_4 has proven to be an effective scale in signifying a frozen concentration field, as compared with a velocity field, or vice versa.

CONCLUSIONS

Dimensionless effective diffusivity, the representative quantity for enhancement of axial dispersion in a laminar oscillating flow, is evaluated by the dimensionless frequency, the dimensionless amplitude and the Schmidt number for the case of a uniform rectangular duct in this study. The results in a rectangular duct share the same physics as those in a circular tube.

Based on a characteristic time scale analysis, the new proper parameters ε_3 and ε_4 are introduced for bridging the discrepancies between mathematical results and physical understanding. Specifically, the augmentation of axial dispersion vanishes for both very high and very low values of ε_3 with constant ε_4. These results mean that the axial diffusion time should be taken into account in addition to the radial diffusion time and the oscillation period for physical understanding of these phenomena. Hence, ε_3 and ε_4 are shown to be more effective parameters than others.

In view of the characteristic time scales related to the new parameters, it may be concluded that one of the underlying principles governing these phenomena is the radial molecular diffusion due to the axial convective transport of the frozen concentration field.

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APPENDIX

(1) Coefficients a_{mn}, b_{mn}, c_{mn} and d_{mn} in equations (11) and (12) are defined as:

$$a_{mn} = -4\frac{c_2^2\alpha_h^4}{(\pi_m^2+c_1\pi_n^2)^2+c_2^2\alpha_h^4}\beta_{mn}$$

$$b_{mn} = -4\frac{(\pi_m^2+c_1\pi_n^2)c_2\alpha_h^2}{(\pi_m^2+c_1\pi_n^2)^2+c_2^2\alpha_h^4}\beta_{mn}$$

$$c_{mn} = \frac{\Gamma_{mn}c_3}{\pi_{mn}^2}(c_3A_{mn}-\Pi_{mn}B_{mn})$$

$$c_{mn} = \frac{\Gamma_{mn}c_3}{\pi_{mn}^2}(c_3A_{mn}-\Pi_{mn}B_{mn})$$

where

$$\pi_m = \frac{2m-1}{2}\pi \quad \beta_{mn} = \frac{(-1)^{m+n}}{\pi_m\pi_n}$$

$$c_3 = c_2\sigma\alpha_h^2 \quad \Pi_m = m\pi \quad \Pi_{mn} = \Pi_m^2+c_1\Pi_n^2$$

$$A_{mn} = \sum_{r,s=1} E_{rsmn}a_{rs} \quad B_{mn} = \sum_{r,s=1} E_{rsmn}b_{rs}$$

$$E_{rsmn} = \frac{\pi_r\pi_s}{(\pi_r^2-\Pi_m^2)(\pi_s^2-\Pi_n^2)}(-1)^{r+s+m+n}.$$

The variable Γ_{mn} is given by

$$\Gamma_{mn} = 4 \quad \text{for } m \neq 0 \quad \text{and} \quad n \neq 0$$

$$\Gamma_{mn} = 2 \quad \text{for } m \neq 0 \quad \text{and} \quad n = 0$$

$$\text{for } m = 0 \quad \text{and} \quad n \neq 0$$

$$\Gamma_{mn} = 1 \quad \text{for } m = 0 \quad \text{and} \quad n = 0.$$

(2) Definition of tidal volume V in equation (13):

$$V = A\chi = A\frac{2P}{\rho\omega^2}\left|\iint_\Omega (F-1)\,d\xi\,d\eta\right|.$$